Comment on 'Decomposition of pure states of a quantum register'

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(10 November 2000)

I. Raptis and R. Zapatrin in the quant-ph/0010104 show possibility to express general state of l-qubits quantum register as sum at most $2^l - l$ product states. In the comment is suggested more simple construction with possibility of generalization for decomposition of tensor product of Hilbert spaces with arbitrary dimension n (here simplicial complexes used in the article mentioned above would not be applied directly). In this case it is decomposition with $n^l - (n^2 - n)l/2$ product states.

PACS numbers: 03.67

I. DECOMPOSITION OF QUBITS AND QU*N*ITS

Result of [1] is proved with using simplicial complexes, etc., but it can be shown also with less special constructions. Such simplification may be useful because it makes possible to apply similar construction not only for qubit spaces $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$, but also for 'qu$n$it' spaces $\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$. Here is represented only short introduction to construction with arbitrary n, because *the result is already known and described in* [2].

The result of [1] is related with possibility of exclusion by local unitary transformations of l terms with one 'unit' like $|100\ldots0\rangle$, $|010\ldots0\rangle$, etc., from general l-qubit state with 2^l terms.

To show it, let us consider some state $|h\rangle = \alpha|00\ldots0\rangle + \alpha_1|10\ldots0\rangle + \cdots$, where α may be zero. It is possible with using a local unitary transformation in first qubit $U_1\colon(\alpha, \alpha_1) \to (\alpha', 0)$ to produce state $|h'\rangle = \alpha'|00\ldots0\rangle + 0|10\ldots0\rangle + \cdots$, where $|\alpha'|^2 = |\alpha|^2 + |\alpha_1|^2$. Next, because $|h'\rangle = \alpha'|00\ldots0\rangle + \alpha_2|01\ldots0\rangle + \cdots$, it is possible to eliminate α_2 by a local unitary transformation of second qubit and with l similar steps we can produce some state $|h^{(1)}\rangle$. If $|h\rangle$ is product state, then $|h^{(1)}\rangle$ has only one nonzero term, but in general case it is possible to produce sum with no more than $2^l - l$ terms only by iterating the process: $|\tilde{h}\rangle = \lim_{N\to\infty} |h^{(N)}\rangle$.

To check convergence let us consider a slightly different algorithm, when N'th step is elimination of term with one unit in position i_N where coefficient $\alpha^{(N)}_{i_N} = \alpha^{(N)}_{\max}$ is maximal between l similar terms. Absolute value of coefficient $\alpha^{(N)}$ of term $|00\ldots0\rangle$ is always limited $|\alpha^{(N)}| \le |h| \equiv 1$ and meets an equation: $|\alpha^{(N)}|^2 = |\alpha|^2 + \sum_{K=1}^{N} |\alpha^{(K)}_{\max}|^2$. So $\sum_{N=1}^{\infty} |\alpha^{(N)}_{\max}|^2 = |\tilde{\alpha}|^2 - |\alpha|^2$ and then $\lim\limits_{N\to\infty} \alpha^{(N)}_{\max} = 0$.

To generalize the method to arbitrary $n > 2$ ('qu$n$it' [3]) let us consider first $n = 3$ ('qutrit' [4]). If we start with some state $|h\rangle = \alpha|00\ldots0\rangle + \alpha_1|10\ldots0\rangle + \beta_1|20\ldots0\rangle + \cdots$, it is possible with using a local unitary $SU(3)$ transformation in first qutrit $U_1\colon(\alpha, \alpha_1, \beta_1) \to (\alpha', 0, 0)$ to produce state $|h'\rangle = \alpha'|00\ldots0\rangle + 0|10\ldots0\rangle + 0|20\ldots0\rangle + \cdots$ with $|\alpha'|^2 = |\alpha|^2 + |\alpha_1|^2 + |\beta_1|^2$ and algorithms similar with discussed above may eliminate $2l$ components with one '1' or '2' and $l-1$ '0'.

But in the case it is not all components appropriate for exclusion. Let us write state $|g\rangle$ without the $2l$ term as: $|g\rangle = \gamma|11\ldots1\rangle + \gamma_1|21\ldots1\rangle + \cdots$. Because it is possible to transform qutrit space without change of component $|0\rangle$, i.e., $U_1\colon(\alpha, \gamma, \gamma_1) \to (\alpha, \gamma', 0)$ we can exclude all l components with one '2' and $l-1$ '1' and so it is possible to have no more than $3^l - 3l$ components.

The similar process is possible to apply to arbitrary qu$n$it $\mathbb{C}^n$. First, we can exclude $(n-1)l$ terms with one '1' and $l-1$ '0'. Second, $(n-2)l$ terms with one '2' and $l-1$ '1'. After $n-1$ iterations we can neglect $((n-1) + (n-2) + \cdots + 1)l = n(n-1)l/2$ terms and so we have no more than

$$n^l - \tfrac{1}{2}n(n-1)l$$

components. For $l = 2$ we have $n^2 - n(n-1)2/2 = n$ terms of usual Schmidt decomposition.

ACKNOWLEDGEMENTS

Author is grateful to R. Zapatrin and C. Zalka for explanations and comments. Many thanks to A. Sudbery for reference to the work [2].

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